Inverse Optimization For Multi-objective Linear Programming

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Inverse Optimization

- Some parameters are unknown (e.g., cost function, right hand side)

- Some extra information available (e.g., optimal solution)

$$min \{cx \mid x \in S\}$$

$$S = \{x \in \mathbb{R}^n \mid Ax \geq b\}$$

$$\begin{aligned} min \ & \|c - \hat{c}\|_p \\ s.t. \ & \hat{x} \in S_o(\hat{c}), \\ & \hat{c} \in \mathbb{R}^n. \end{aligned}$$

- Smallest changes in c to make $\hat{x}$ an optimal solution?

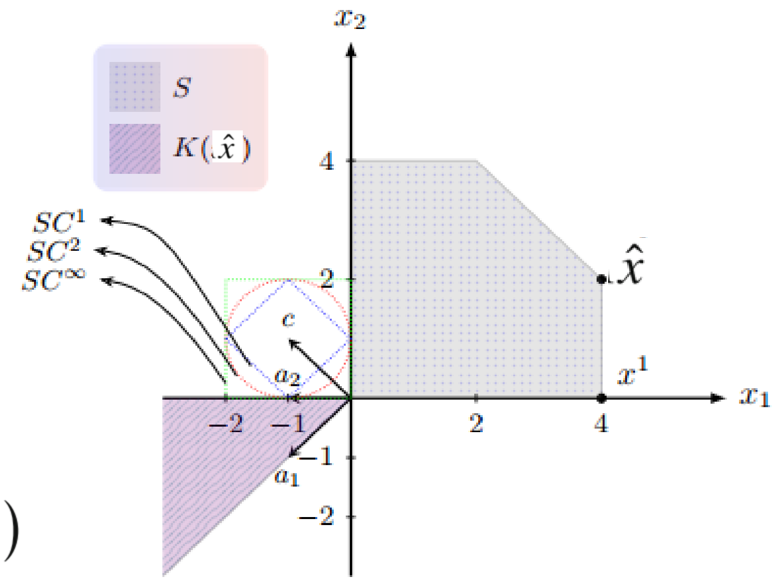
Inverse Linear Programming

❖ KKT Conditions

$$\min \|c - \hat{c}\|_p$$
$$\text{s.t.}\quad y(A\hat{x} - b) = 0, \quad \text{(complementary slackness)},$$
$$yA = \hat{c}, \quad \text{(dual feasibility)},$$
$$y \geq 0, \quad \text{(dual feasibility)},$$
$$\hat{c} \in \mathbb{R}^n.$$

$$\min \|c - \hat{c}\|_p$$
$$\text{s.t.}\quad \hat{c} \in \hat{K},$$
$$\hat{c} \in \mathbb{R}^n.$$

$$I(\hat{x}) = \{i \mid a_i\hat{x} = b_i\} \quad \hat{K} = cone\left(\{a_i \mid i \in I(\hat{x})\}\right)$$



Inverse Multi-objective Linear Programming

$$\begin{aligned} &\min \|C - \hat{C}\| \\ &s.t. \quad \hat{x} \in S_{we}(\hat{C}), \\ &\qquad \hat{C} \in \mathbb{R}^{k \times n}, \end{aligned}$$

⇔

$$\begin{aligned} &\min \|C - \hat{C}\| \\ &s.t. \quad w\hat{C} \in \hat{K}, \\ &\qquad w \in W, \\ &\qquad \hat{C} \in \mathbb{R}^{k \times n}. \end{aligned}$$

Chan, T.C.Y., Craig, T., Lee, T., Sharpe, M.B.: Generalized inverse multiobjective optimization with application to cancer therapy. Oper. Res. **62**(3), 680–695 (2014). https://doi.org/10.1287/opre.2014.1267

Chan, T.C.Y., Lee, T.: Trade-off preservation in inverse multi-objective convex optimization. Eur. J. Oper. Res. **270**(1), 25–39 (2018). https://doi.org/10.1016/j.ejor.2018.02.045

Non-convexity issue

$$\begin{aligned} min \ & \|C - \hat{C}\| \\ s.t. \ & w\hat{C} \in \hat{K}, \\ & w \in W, \\ & \hat{C} \in \mathbb{R}^{k \times n}. \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} \min \ \rho = & \sum_{i=1}^{k} \|c_i - \hat{c}_i\|_p \\ s.t. \ & \sum_{i=1}^{k} w_i \hat{c}_i - \sum_{r \in I(\hat{x})} \beta_r a_r = 0, \\ & \sum_{i=1}^{k} w_i = 1, \\ & w_i \geq 0, \qquad i = 1, \ldots, k, \\ & \beta_r \geq 0, \qquad r \in I(\hat{x}), \\ & \hat{c}_i \in \mathbb{R}^n, \qquad i = 1, \ldots, k. \end{aligned}$$

Main Results

Theorem 2 *For* IMOLP$(C, \hat{x})$ (4) *if* $d(conv(C), \hat{K}) > 0$, *then*

- *(i)* IMOLP$(C, \hat{x})$ (4) *has an optimal solution* $\hat{C}^*$ *for which* $\hat{c}_i^* = c_i$ *for* $i \in \{1, \ldots, k\}$, *and* $i \neq j$.
- *(ii)* $d(conv(C), \hat{K})$ *provides a lower bound for the optimal value of* IMOLP$(C, \hat{x})$ (4), *i.e.,* $\rho^* \geq d(conv(C), \hat{K})$.

$$
\begin{aligned}
\min\ & \|c_j - \hat{c}_j\|_p \\
s.t.\ & w_j \hat{c}_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} w_i c_i - \sum_{r \in I(\hat{x})} \beta_r a_r = 0, \\
& \sum_{i=1}^{k} w_i = 1, \\
& w_i \geq 0, \quad i = 1, \ldots, k, \\
& \beta_r \geq 0, \quad r \in I(\hat{x}), \\
& \hat{c}_j \in \mathbb{R}^n.
\end{aligned}
$$

Non-convex

$$
\begin{aligned}
\rho_j^* = \min\ & \|c_j - \hat{c}_j\|_p \\
s.t.\ & \hat{c}_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} \lambda_i c_i - \sum_{r \in I(\hat{x})} \alpha_r a_r = 0, \\
& \lambda_i \geq 0, \quad i = 1, \ldots, k, i \neq j, \\
& \alpha_r \geq 0, \quad r \in I(\hat{x}), \\
& \hat{c}_j \in \mathbb{R}^n.
\end{aligned}
$$

Convex

Geometric Interpretation

$$\rho_j^* = \min \|c_j - \hat{c}_j\|_p$$

$$s.t. \;\; \hat{c}_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} \lambda_i c_i - \sum_{r \in I(\hat{x})} \alpha_r a_r = 0,$$

$$\lambda_i \geq 0, \qquad i = 1, \ldots, k, i \neq j,$$

$$\alpha_r \geq 0, \qquad r \in I(\hat{x}),$$

$$\hat{c}_j \in \mathbb{R}^n.$$

$$\{\{a_r\}_{r \in I(\hat{x})}, \{-c_i\}_{i=1,\neq j}^{k}\}$$

$$d(c_j, K_j) = min \; \|c_j - \hat{c}_j\|_p$$

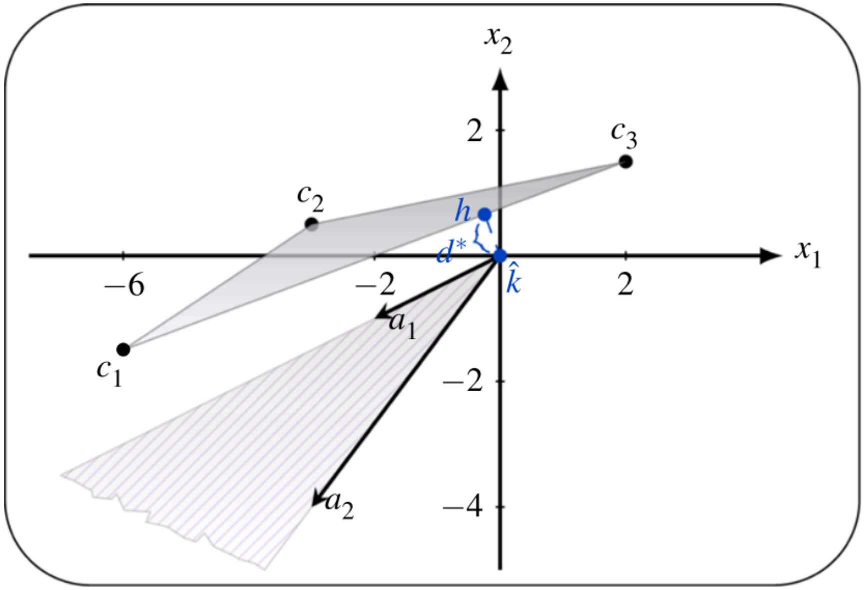
$$s.t. \;\; \hat{c}_j \in K_j,$$

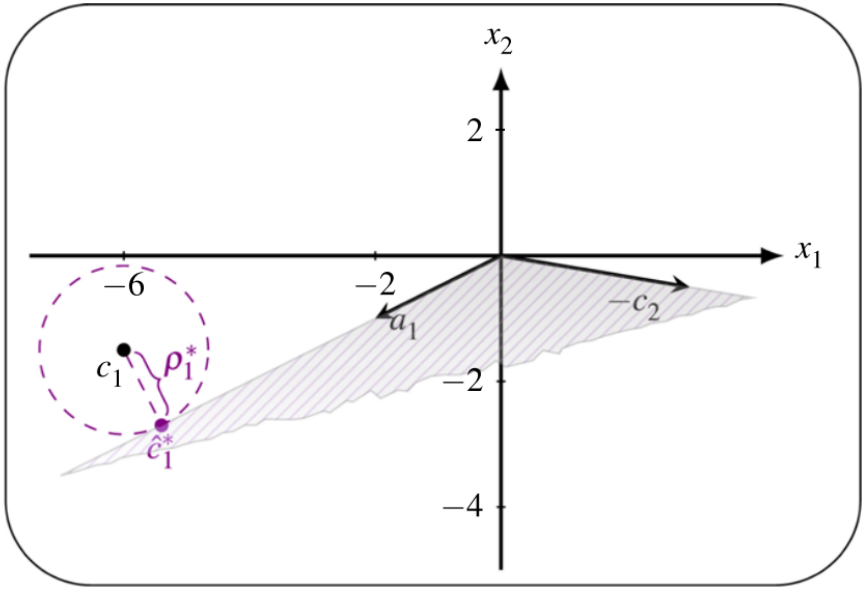
$$\hat{c}_j \in \mathbb{R}^n.$$

Simple Example

$$min\,\{Cx \mid Ax \geq b\} \qquad \hat{x} = (8, 7) \qquad I(\hat{x}) = \{1, 2\}$$

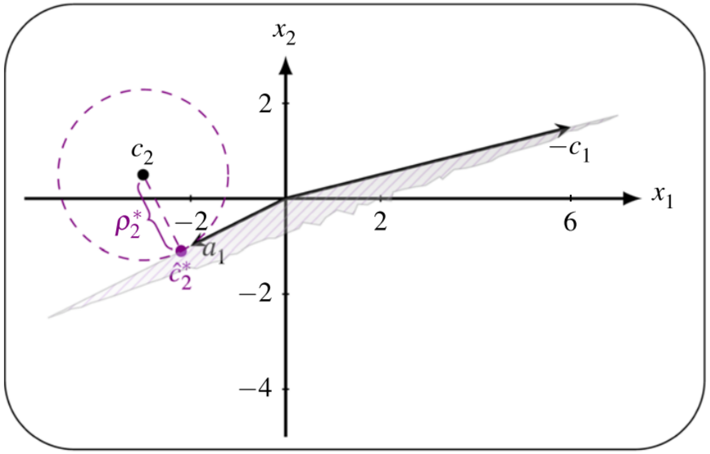
$$C = \begin{pmatrix} -6 & -1.5 \\ -3 & 0.5 \\ 2 & 1.5 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & -1 \\ -3 & -4 \\ -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} -23 \\ -52 \\ -10 \\ -10 \\ 0 \\ 0 \end{pmatrix}$$

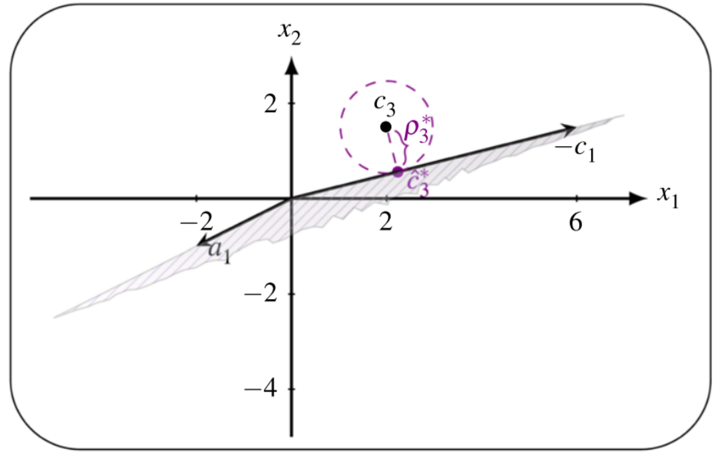


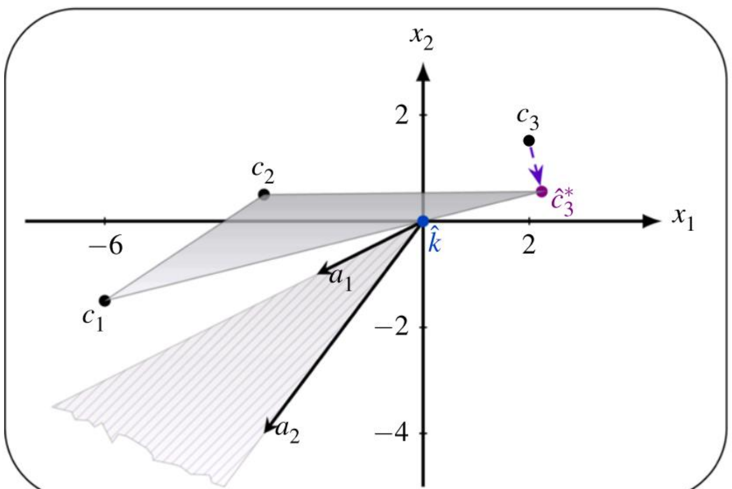


$$K_1 = cone\left(\{a_1, a_2, -c_2, -c_3\}\right) = cone\left(\{a_1, -c_2\}\right)$$

Simple Example





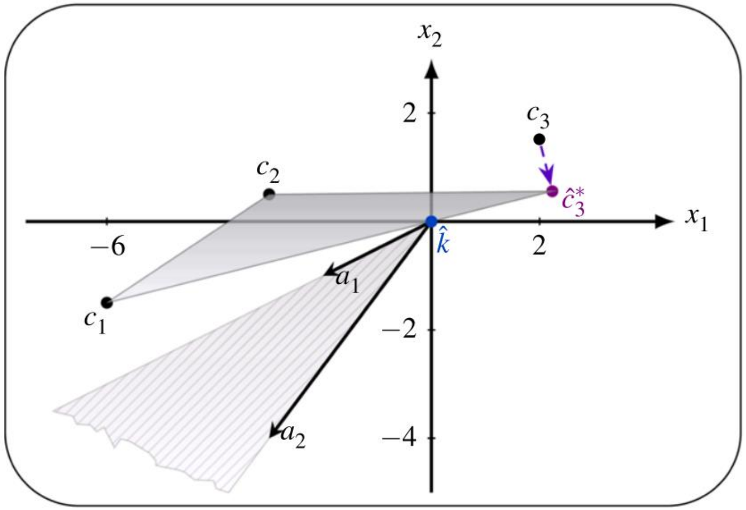


(e) The result of the algorithm. If c_3 moves to $\hat{c}_3^*$, then $conv(\hat{C}^*)$ intersects $\hat{K}$, making $\hat{x}$ a weakly efficient point.

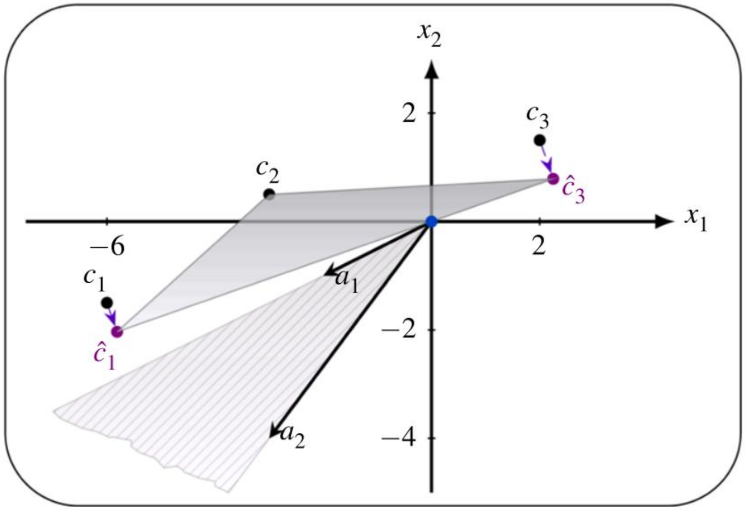
Future Challenge

❖ **What if there is a constraint on how much C can be changed?**

Counter example



(e) The result of the algorithm. If c_3 moves to $\hat{c}_3^*$, then $conv(\hat{C}^*)$ intersects $\hat{K}$, making $\hat{x}$ a weakly efficient point.



(f) Moving c_1 and c_3 simultaneously to turn $\hat{x}$ into a weakly efficient point.

Summary

- ❖Generalized inverse linear programming to multi-objective
- ❖Problem is originally non-convex
- ❖Can be formulated as convex using the special characteristics
- ❖ Doesn't work if the amount of change on C is constrained
- ❖ Application

Optimization Letters
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ORIGINAL PAPER

Inverse optimization for multi-objective linear programming

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